Jeff Adams lectures at Atlas Conference, Summer 2010, in Salt Lake City, Utah

July 28, 2010

1 Lecture on Character Theory on 07/18/10 at 9AM

Overarching questions:

1) What did David Vogan mean when he said we computed the character table of E_8?

2) What did Gregg Zuckerman mean when he said that the characters of unitary representations are simpler than arbitrary ones?

3) What did Jim Arthur mean when he told us to study unipotent representations? In particular, what can we say about the characters of unipotent representations?

I. Basics: (reference is Knapp's book containing the word "Overview" in the title)

Let $G(\mathbb{R})$ be a very general real reductive group. Let $f \in C_c^\infty(G(\mathbb{R}))$ and (π, V) an admissible representation of $G(\mathbb{R})$ of finite length. Define $\pi(f)v := \int_{G(\mathbb{R})} f(g)\pi(g)vdg$, where we have chosen a Haar measure dg on G.

Definition 1.1. We define the *distribution character* of π by

$$\theta_\pi(f) = Tr(\pi(f))$$

Notes:

1)$\pi(f)$ is a trace class operator, so $Tr(\pi(f))$ is well-defined.

2) If $0 \to \pi_1 \to \pi_2 \to \pi_3 \to 0$ is exact, then $\theta_{\pi_2} = \theta_{\pi_1} + \theta_{\pi_3}$.

Therefore, θ is really a function on the Grothendieck group of representations.

Theorem 1.2. *(Harish-Chandra) If π, π' are irreducible, then $\pi \cong \pi'$ if and only if $\theta_\pi = \theta_{\pi'}$.*

Note : What we mean by $\pi \cong \pi'$ is that π is *infinitesimally equivalent* to π'.

Definition 1.3. $G(\mathbb{R})_{rss} :=$ regular semisimple elements of $G(\mathbb{R}) = \{g \in G(\mathbb{R}) : g$ is semisimple, $g \in H(\mathbb{R}) =$ Cartan, $e^\alpha(g) \neq 1\ \forall$ roots $\alpha\}$.

$G(\mathbb{R})$ acts on $G(\mathbb{R})_{rss}$ by conjugation, and $G(\mathbb{R})_{rss}$ is open and dense in $G(\mathbb{R})$.

Theorem 1.4. *(Harish-Chandra) Let π be admissible, finite length representation of $G(\mathbb{R})$. Then there exists a unique continuous function F_π on $G(\mathbb{R})_{rss}$ which is conjugation invariant, such that*

$$\theta_\pi(f) = \int_{G(\mathbb{R})} f(g) F_\pi(g) dg$$

Corollary 1.5. *Let π, π' be irreducible. Then π is infinitesimally equivalent to π' if and only if $F_\pi = F_{\pi'}$.*

Notes:

1) F_π is independent of the choice of Haar measure dg. However, θ_π does depend on the choice of Haar measure dg.

2) F_π has possible singularities as you approach $G(\mathbb{R}) \setminus G(\mathbb{R})_{rss}$.

From now on, we will identify θ_π with F_π, so we will view θ_π as a function on $G(\mathbb{R}_{rss}$.

Example: "$\theta_\pi(1)''$ = "$dim(\pi)''$.

Sidenote: It is interesting to study singularities of θ_π near 1.

Note: $g \in G(\mathbb{R})_{rss}$ implies that g is contained in a unique Cartan subgroup. Moreover, $\{\text{Cartans}\}/\sim \; = \; \{H_1(\mathbb{R}), \cdots, H_n(\mathbb{R})\}$ is a finite set, where $\sim$ denotes conjugation. So θ_π is determined by its restriction to the union of the $H_i(\mathbb{R})/W(G(\mathbb{R}), H_i(\mathbb{R}))$, where $W(G(\mathbb{R}), H_i(\mathbb{R})) := N_{G(\mathbb{R})}(H_i(\mathbb{R}))/H_i(\mathbb{R})$ is the real Weyl group.

Definition 1.6. If $\mathfrak{g}$ is a complex reductive Lie algebra, then a *Cartan subalgebra* is a maximal, abelian, semisimple subalgebra of $\mathfrak{g}$.

Definition 1.7. A *Cartan subgroup* of $G(\mathbb{R})$ is $Z_{G(\mathbb{R})}(\mathfrak{h})$, where $\mathfrak{h}$ is a Cartan subalgebra defined over $\mathbb{R}$ inside $\mathfrak{g} = Lie(G(\mathbb{R})) \otimes \mathbb{C}$.

Notes :

1) If $G(\mathbb{R})$ is real reductive, then there is a map $\phi : G(\mathbb{R}) \to Ad(\mathfrak{g})$.

2) In Atlas, $G(\mathbb{C}) = G =$ complex connected reductive group, and $G(\mathbb{R}) = G(\mathbb{C})^\sigma =$ real points.

Example: Consider $SO(p,q)$. This group is considered in Atlas. It's a real, reductive, algebraic group. $SO_e(p,q) \subset SO(p,q)$ is not considered in Atlas, when $pq \neq 0$.

Example: $\mathbb{R}^*$ is considered in Atlas, but $\mathbb{R}_{>0}$ is not considered in Atlas. So Atlas excludes some groups.

Serious Example: $G(\mathbb{C}) =$ simple, simply connected. Let $G(\mathbb{R})$ be its split real form. Fact: $\pi_1(G(\mathbb{R})) \neq 1$, even though $\pi_1(G(\mathbb{C})) = 1$. The group $\widetilde{G(\mathbb{R})}$, the simply connected cover, is not considered in Atlas. For example, $\widetilde{SL(2,\mathbb{R})}$ is not in Atlas.

Example: Let $G(\mathbb{R}) = GL(n,\mathbb{R})$. Suppose $a+2b = n$. Then it's possible to embed $(\mathbb{R}^*)^a \times (\mathbb{C}^*)^b$ into $GL(n,\mathbb{R})$ where $y \in (\mathbb{R}^*)^a$ goes to

$$\begin{pmatrix} y & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & y \end{pmatrix}$$

and where

$$\alpha + i\beta \mapsto \begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots \end{pmatrix}$$

Fact : $\{Cartans\ in\ GL(n,\mathbb{R})\}/\sim \;=\; \{(\mathbb{R}^*)^a \times (\mathbb{C}^*)^b : a+2b = n\}$

Exercise: What are the Cartans (up to conjugacy) in $SL(n,\mathbb{R})$.
(a) How many are there?
(b) Describe $H_i(\mathbb{R})$ as a real torus. (It's a fact that any real torus is isomorphic to a product of circles, $\mathbb{R}^*$'s, and $\mathbb{C}^*$'s.)

Back to distribution characters.
1. If G is compact and connected, θ_π is given by the Weyl character formula.

Example: $SL(2,\mathbb{R})$. Let A be the diagonal torus and T the torus

$$\left\{ e^{i\theta} = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\} \cong S^1.$$

Suppose χ is a character of $\mathbb{R}^*$. Let $\pi(\chi) = Ind_B^G(\chi)$ be the corresponding principal series.

Then $\theta_{\pi(\chi)}(g) = 0$ if $g = e^{i\theta}$ and $\theta_{\pi(\chi)}(g) = \frac{\chi(x)+\chi(1/x)}{|x-1/x|}$ when $g = x \in A$ (we are assuming g is regular).

Theorem 1.8. *(Harish-Chandra) Let $G(\mathbb{C})$ be simply connected, SEMIsimple, and let $T(\mathbb{R}) \subset G(\mathbb{R})$ be a compact Cartan. Suppose Λ is a character of $T(\mathbb{R})$ such that Λ is regular, i.e. $< d\Lambda, \alpha^\vee > \neq 0$ for all roots α. Then there exists a unique discrete series representation $\pi(\Lambda) = \pi$ such that $\theta_\pi(t) = \frac{(-1)^q}{D(\Delta^+,t)} \sum_{w \in W(G(\mathbb{R}),T(\mathbb{R}))} \mathrm{sgn}(w)(w\Lambda)(t)$ where $\Delta^+ = \{\alpha :< d\Lambda, \alpha^\vee >> 0\}$, $D(\Delta^+,t) = \prod_{\alpha\in\Delta^+}(1-e^{-\alpha}(t))e^\rho(t)$, and $q = \frac{1}{2}dim(G(\mathbb{R})/K(\mathbb{R}))$.*

Notes:
1) $G(\mathbb{C})$ simply connected and semisimple implies that $e^\rho(t)$ is well-defined.

2) If $G(\mathbb{C})$ is reductive, then if the derived group is simply connected, this doesn't necessarily imply that $e^\rho(t)$ is well-defined.

3) $\pi(\Lambda) \cong \pi(\Lambda')$ if and only if $\Lambda = w\Lambda'$, where $w \in W(G(\mathbb{R}), T(\mathbb{R}))$. So, the discrete series up to isomorphism are parameterized by $\{(T(\mathbb{R}), \Lambda)\}/G(\mathbb{R})$. Finally, every discrete series is isomorphic to $\pi(\Lambda)$, for some Λ.

Example: $SL(2,\mathbb{R})$. Let Λ_k be the character $\Lambda(e^{i\theta}) = e^{ik\theta}$, where $k \neq 0$. Then $\theta_{\pi(\Lambda_k)}(e^{i\theta}) = \frac{-e^{ik\theta}}{e^{i\theta}-e^{-i\theta}}$ if $k > 0$, and $\theta_{\pi(\Lambda_k)}(e^{i\theta}) = \frac{e^{ik\theta}}{e^{i\theta}-e^{-i\theta}}$ if $k < 0$. Well, what's the formula for $\theta_{\pi(\Lambda_k)}(x)$ where x is an element of the split Cartan? Well, the answer is as follows: First,

Definition 1.9. Let $\epsilon = \pm$ and let $k \geq 1$. Define $\pi_k^+ := \pi(\Lambda_k)$ and $\pi_k^- := \pi(\Lambda_{-k})$ so that we can talk about π_k^ϵ.

Then $\theta_{\pi_k^\pm}(x) = \frac{x^{-k}}{x-1/x}$ when $|x| > 1$, and $\theta_{\pi_k^\pm}(x) = \frac{-x^k}{x-1/x}$ when $|x| < 1$ (where x is an element of the split Cartan).

2 Lecture on Character Theory on 07/19/10 at 9AM

Recall from the handout we have various representations of $SL(2,\mathbb{R})$. The discrete series $\pi_k^\pm$, the principal series $\pi(\chi(\nu,\epsilon))$, and the finite dimensional representations F_k. You can see from the characters of these representations, that

$$\pi_k^+ + \pi_k^- + F_k = \pi(\chi(k,(-1)^{k+1}))$$

on the level of characters. Therefore, if k is an integer, then $\pi(\chi(k,(-1)^{k+1}))$ has π_k^+, π_k^-, F_k in its composition series. Moreover, if you evaluate both sides of this equation of characters on the compact Cartan, you get that since the character of the principal series vanishes on the compact Cartan, you get that

$$\theta_{\pi_k^+}(e^{i\theta}) + \theta_{\pi_k^-}(e^{i\theta}) = -\theta_{F_k}(e^{ik\theta})$$

Note that $\pi(\chi(\nu,\epsilon))$ is irreducible unless

a) $\nu \in \mathbb{Z}$

b) $\epsilon = (-1)^{\nu+1}$.

Sometimes we will write $\pi(\nu,\epsilon)$ instead of $\pi(\chi(\nu,\epsilon))$. If you fire up Atlas, and you look at the representations of $SL(2,\mathbb{R})$, you have to choose a dual group. If you choose ${}^\vee G = PGL(2,\mathbb{R})$, you get 3 representations : $\pi_1^+, \pi_1^-, \pi(1,1)$. If you choose the dual group to be $SO(3)$, you get 1 representation : $\pi(1,-1)$. This is at infinitesimal character 1. If you're at inifinitesimal character 2, you get a block $\{\pi_2^+, \pi_2^-, \pi(2,-1)\}$, and $\pi(2,1)$.

Recall that if $G(\mathbb{C})$ is simply connected and semisimple, and $\Lambda \in \widehat{T(\mathbb{R})}$ where $T(\mathbb{R})$ is a compact Cartan, we gave a formula for the discrete series representation $\pi(\Lambda)$. The denominator had the denominator $e^\rho(t)\prod_{\alpha\in\Delta^+}(1-e^{-\alpha}(t))$. We want to say what the character formulas are for more general groups.

So we need to talk about Weyl denominators.

Definition 2.1. Let $G = G(\mathbb{C})$ be a complex group, $H = H(\mathbb{C})$ a Cartan in G. Suppose you have a set of positive roots Δ^+. Define $D^0(h) = \prod_{\alpha\in\Delta^+}(1 - e^{-\alpha}(h))$, and $|e^\rho(h)| := |e^{2\rho}(h)|^{1/2}$, and $|D(h)| := |D^0(h)||e^\rho(h)|$. Then "$|D(h)| = |\prod_\alpha(e^{\alpha/2}(h) - e^{-\alpha/2}(h)|$". Then $|D(h)|$ is the *absolute value of the Weyl denominator*.

We want to define $D(h) := D^0(h)e^\rho(h)$. Unfortunately, $e^\rho(h)$ is not well-defined necessarily.

Definition 2.2. Suppose we have a Δ^+, ρ. Define H_ρ as the pullback of the two maps $H \xrightarrow{e^{2\rho}} \mathbb{C}^*$ and $\mathbb{C}^* \xrightarrow{z^2} \mathbb{C}^*$.

Explicitly, $H_\rho = \{(h, z) \in H \times \mathbb{C}^* : e^{2\rho}(h) = z^2\}$.

Note: $H_\rho \cong H_{w\rho}$ by the map $(h, z) \mapsto (h, e^{w\rho-\rho}(h)z)$.

Definition 2.3. Define $\tilde{H} :=$ inverse limit of the $H_{w\rho}$ over all w, via the previous maps.

Then we have a canonical map $\tilde{H} \to H$, and we canonically have $\tilde{H} \cong H_\rho$ for any choice of ρ.

Note: $H_\rho \cong H \times \mathbb{Z}/2\mathbb{Z}$ if and only if ρ exponentiates to H.

Definition 2.4. Let $G(\mathbb{R})$ be a real connected reductive group, and a Cartan $H(\mathbb{R})$. Then we can define $H(\mathbb{R})_\rho$ and $\widetilde{H(\mathbb{R})}$.

Note that $D^0(h)^2e^{2\rho}(h) \in \mathbb{R}^*$, for $h \in H(\mathbb{R})$.

Definition 2.5. $D(\tilde{h}) := D^0(h)e^\rho(\tilde{h})$, for $\tilde{h} \in H_\rho$, and h is the image of $\tilde{h}$ under the natural projection.

Definition 2.6. Let $G(\mathbb{C})$ be connected, complex, reductive algebraic group, with real points $G(\mathbb{R})$, and supposed it has a compact mod center Cartan $T(\mathbb{R})$. Let Λ be a genuine character of $\widetilde{T(\mathbb{R})}$ that is regular (i.e. $< d\Lambda, \alpha^\vee >\neq 0\ \forall\alpha$, where $\alpha \in \Delta^+$, and Δ is the set of roots of G with respect to T. Then there exists a unique (relative) discrete series representation $\pi(\Lambda)$ such that

$$\theta_{\pi(\Lambda)}(t) = \frac{(-1)^{q_G}}{D(\tilde{t})}\sum_{w\in W(G(\mathbb{R}),T(\mathbb{R}))} sgn(w)(w\Lambda)(\tilde{t})$$

where $q_G := (-1)^{\frac{1}{2}dim(G_d(\mathbb{R})/K_d(\mathbb{R}))}$ and $\Delta^+ = \{\alpha :< d\Lambda, \alpha >> 0\}$, and where G_d, K_d means derived groups. Moreover, $\tilde{t}$ is any element of $\widetilde{T(\mathbb{R})}$ that maps to $t \in T(\mathbb{R})$.

Let $G(\mathbb{R})$ be a real group with a Cartan $H(\mathbb{R})$, and a Cartan involution θ. Assume that $H(\mathbb{R})$ is θ-stable.

Definition 2.7. If α is a root, then α is real if $\overline{\alpha} = \alpha$, which is the same thing as saying that $\theta(\alpha) = -\alpha$, and it's imaginary if $\overline{\alpha} = -\alpha$, which is the same thing as saying that $\theta(\alpha) = \alpha$. It is called complex otherwise.

Choose ψ to be a set of positive real roots inside of the set of real roots Δ_r. The set of real roots Δ_r and the set of imaginary roots Δ_i are both root systems. The set of imaginary roots are the fixed points of Δ under θ, and the set of real roots are the fixed points of Δ under $-\theta$.

Definition 2.8. $H(\mathbb{R})_+ := H(\mathbb{R})_+(\psi) := \{h \in H(\mathbb{R}) : |e^\alpha(h)| > 1 \ \forall \alpha \in \psi\}$.

Every regular element of $H(\mathbb{R})$ is conjugate to some element of $H(\mathbb{R})_+$, since $s_\alpha \in W(G(\mathbb{R}), H(\mathbb{R}))$ if $\alpha \in \Delta_r$.

Example: Consider $SL(2, \mathbb{R})$ and $H(\mathbb{R}) = \mathbb{R}^*$, the split Cartan. Let ψ be the root α which satisfies $\alpha(x) = x^2$. Then $H(\mathbb{R})_+ = \{x : |x| > 1\}$. If we chose the opposite root, we would have gotten $H(\mathbb{R})_+ = \{x : |x| < 1\}$. The Weyl group acts as $x \mapsto 1/x$.

Theorem 2.9. *(Harish-Chandra) Let G be real reductive, H a real Cartan. Fix a set of positive roots Δ^+. Let $\psi = \Delta^+ \cap \Delta_r$. Suppose π is an admissible, finite length, representation with infinitesimal $\lambda \in \mathfrak{h}^*$. Then*

$$\theta_\pi(h) = \sum_\Lambda \frac{a(\pi, \Delta^+, \Lambda)\Lambda(\tilde{h})}{D(\tilde{h})}$$

where $h \in H(\mathbb{R})_+$, where the sum is over all genuine characters Λ of $\widetilde{H(\mathbb{R})}$, $d\Lambda$ is conjugate to λ by $W(G(\mathbb{C}), H(\mathbb{C}))$. $\tilde{h}$ is any element that maps to $h \in H(\mathbb{R})_+$. Moreover, $a(\pi, \Delta^+, \Lambda) \in \mathbb{Z}$.

Note: This formula is valid on all of $H(\mathbb{R})_+$, which is a big set, but not necessarily on all of $H(\mathbb{R})$. Moreover, the number of terms in the sum is at most $|W||H(\mathbb{R})/H(\mathbb{R})^0|$, where W is the Weyl group.

Exercise: Find a virtual character π of $SL(2, \mathbb{R})$ that vanishes on the split Cartan.

3 Lecture on Character Theory on 07/20/10 at 9AM

Recall that we had $D^0(h) = \prod_{\alpha \in \Delta^+}(1 - e^{-\alpha}(h))$, $|D(h)| = |D^0(h)||e^\rho(h)|$, and $D(\tilde{h}) = D^0(h)e^\rho(\tilde{h})$.

Lemma 3.1. *1) $|D(\tilde{h})| = |D(h)|$.*
2) $D(\tilde{h})^2 \in \mathbb{R}^*$.
3) Therefore, $D(\tilde{h}) = |D(h)|\gamma(\tilde{h})$, with $\gamma(\tilde{h})^4 = 1$.

Recall from last time that we had the character formula

$$\theta_\pi(h) = \sum_\Lambda \frac{a(\pi, \Delta^+, \Lambda)\Lambda(\tilde{h})}{D(\tilde{h})}$$

on $H(\mathbb{R})_+$.

The problem is to compute the $a(\pi, \Delta^+, \Lambda)$. If you compute these for all H_i, then you know θ_π, where $H_1, ..., H_n$ are the representatives Cartans mod conjugacy.

Let's talk about $SL(2, \mathbb{R})$, and infinitesimal character $\lambda = 1$, which is ρ, in the usual coordinates. Consider the split Cartan $\mathbb{R}^*$. Since the group is simply connected, we don't have to consider ρ-covers, so we can work on $H(\mathbb{R})$. What are the Λ that are allowed at this infinitesimal character? Well, the characters with differential equal to 1 are the characters $\Lambda(x) = x, |x|, 1/x$, or $1/|x|$. Then

$\theta_\pi(x) = \frac{ax+b|x|+c\frac{1}{x}+d\frac{1}{|x|}}{*}$, and you'd like to compute a, b, c, d. For example,

$$\theta_{\pi(\mu)}(x) = \frac{\mu(x)sgn(x) + \mu(x^{-1})sgn(x^{-1})}{*} \quad \forall |x| > 1$$

Moreover,

$$\theta_{\pi_1^+}(x) = \frac{ax + b|x| + c\frac{1}{x} + d\frac{1}{|x|}}{x - 1/x} \quad \forall |x| > 1$$

for some a, b, c, d. It turns out that $a = b = d = 0$.

So the character table for $SL(2, \mathbb{R})$ at infinitesimal character ρ is given by

$\pi(1,+)$	0	0	1	0	1	0
$\pi(1,-)$	0	0	0	1	0	1
π_1^+	-1	0	0	0	1	0
π_1^-	0	1	0	0	1	0

where the second through seventh columns give you the coefficients of $e^{i\theta}, e^{-i\theta}, x, |x|, 1/x, 1/|x|$, respectively, in the character formulas of $\pi(1,+), \pi(1,-), \pi_1^+$, and π_1^-.

Question: How do you compute the above table? Historically :

1) You compute $\theta_\pi(h)$, where π is a discrete series, and h is in ANY Cartan.

2) The induced character formula gives you the character formulae for induced representations.

These two together, imply the table.

Exercise: Implement various algorithms to compute characters of discrete series, and compare them. For example, compare those of Zuckerman, Kottwitz-MacPherson-Goresky, Schmid, Herb, ...

Definition 3.2. Define the general *standard module* to be $I(H, \Phi, \psi, \Lambda)$, where H is a θ-stable Cartan that is defined over $\mathbb{R}$, Φ is a set of positive imaginary roots, ψ is a set of positive real roots, Λ is a genuine character of $\widetilde{H(\mathbb{R})}$, all satisfying various conditions. This is the data for a general standard module. We will define the representation shortly.

Special case: Suppose $d\Lambda$ is regular and integral, in which case the conditions say that Λ determines Φ and ψ. i.e. Φ and ψ are just the imaginary, real roots, that are positive with respect to Λ. So in this case you just have $I(H, \Lambda)$. We call the pair $(H, \Lambda) =: \gamma$ the standard data.

Definition 3.3. Let $\gamma = (H, \Lambda)$ be a standard data. Define $I(\gamma) = I(H, \Lambda) = Ind_{P(\mathbb{R})}^{G(\mathbb{R})}(I(H, \Lambda_M))$ where $H = TA$, $H(\mathbb{R}) = T(\mathbb{R})A(\mathbb{R})$, $T = H^\theta$, $A = H^{-\theta}$, $M = Cent_G(A)$, $M(\mathbb{R})$, and $P(\mathbb{R}) = M(\mathbb{R})N(\mathbb{R})$. Moreover, $\Lambda_M = \Lambda \otimes \delta$ where δ is a character of $\widetilde{H(\mathbb{R})}$ of order 4. Moreover, Λ lives on $H(\mathbb{R}_\rho$ and Λ_M lives on $H(\mathbb{R})_{\rho_i}$ (they are both genuine characters. Moreover, $\delta = \frac{e^{\rho_r}}{|e^{\rho_r}|}$, and this lives on $H(\mathbb{R})_{\rho_r}$. Then $I(H, \Lambda_M)$ is a relative discrete series of M.

Theorem 3.4. *Consider the case of regular integral infinitesimal character. Let $\gamma = (H, \Lambda)$. Then*

1) $I(\gamma)$ has infinitesimal character $d\Lambda$

2) $I(\gamma)$ has finite length.

3) $I(\gamma)$ has a unique irreducible subquotient $\pi(\gamma)$.

4) $\pi(\gamma) \cong \pi(\gamma')$ if and only if γ is K-conjugate to γ'.

5) Every irreducible representation of infinitesimal character Λ is $\pi(\gamma)$ where $\gamma = (H, \Lambda)$ and $d\Lambda$ is W-conjugate to λ.

6) The set of irreducible representations with infinitesimal character λ are parameterized by pairs (H, Λ) modulo conjugacy.

Comment: The "block" command in atlas computes the set $\{(H, \Lambda)\}/\sim$.

Let's talk about the induced character formula. If $\pi = Ind_{P(\mathbb{R})}^{G(\mathbb{R})}\sigma$, and $H = TA$, then

$$\theta_\pi(h) = |D(\Delta^+, h)|^{-1} \sum_{W(G(\mathbb{R})), H(\mathbb{R})/W(M(\mathbb{R}), H(\mathbb{R}))} |D(\Delta_i^+, wh)|\theta_\sigma(wh) \ \forall h \in H(\mathbb{R})$$

where $H(\mathbb{R})$ is the Cartan coming from $M(\mathbb{R}) \subset P(\mathbb{R})$. Here, the first D is the Weyl denominator for G, and the second D is the Weyl denominator for M.

Corollary 3.5. *We know $\theta_{I(\gamma)}(h)$ for $\gamma = (H, \Lambda)$, for all $h \in H(\mathbb{R})$.*

Next time: We will talk about coherent continuation (via atlas), which will give us a formula for $a(\pi, \Delta^+, \Lambda)$.

4 Lecture on Character Theory on 07/21/10 at 9AM

Recall that we talked about the the modules $I(H, \Lambda)$.

Definition 4.1. Δ^+, Λ are *aligned* if

1) If $\alpha > 0$, α complex, then $\theta(\alpha) < 0$

2) $\{\alpha : im < \lambda, \alpha^\vee >> 0\} \subset \Delta^+$, where $\lambda = d\Lambda$.

3) $\{\alpha : re < \lambda, \alpha^\vee >> 0\} \subset \Delta^+$.

If you fix $H(\mathbb{R})_+$, then Condition 3 is saying that Λ is blowing up as fast as possible on $H(\mathbb{R})_+$.

Recall we gave a formula for $\theta_{I(H,\Lambda)}$. Recall that we also gave a formula

$$a(I(H, \Lambda), \Delta^+, \tau(\Delta^+, w)(w^{-1}\Lambda)) = (-1)^{q_M} sgn(w_i)$$

where $\tau(\Delta^+, w) = sgn(e^{w\rho_r - \rho_r}(h))$.

Lemma 4.2. *If Δ^+, Λ are aligned, then*

$$a(I(H, \Lambda), \Delta^+, \Lambda) = (-1)^{q_M}$$

Lemma 4.3. *(Leading exponents)*

Suppose Λ, Δ^+ are aligned. Then $a(I(H', \Lambda'), \Delta^+, \Lambda) = (-1)^{q_M}$ if (H', Λ') is conjugate to (H, Λ), and is zero otherwise.

Let's go to the KLV picture. Let $\gamma = (H, \Lambda)$. Then we have $I(\gamma), \pi(\gamma)$. Then

$$I(\gamma) = \sum_{\delta} m(\delta, \gamma)\pi(\delta)$$

in the Grothendieck group, for some coefficients m. Moreover,

$$\pi(\gamma) = \sum_{\delta} M(\delta, \gamma)I(\delta)$$

for some coefficients M.

Theorem 4.4. *If Δ^+, Λ are aligned. Then*

$$a(\pi, \Delta^+, \Lambda) = (-1)^{q_H} M(I(H, \Lambda), \pi)$$

Let's now talk about Coherent Continuation.

Let λ be regular integral, and let $\mathcal{M}_\lambda$ be the Grothendieck group of $(\mathfrak{g}, K)$-modules with infinitesimal character λ. This is $\mathbb{Z}[\pi_1, ..., \pi_n] = \mathbb{Z}[I_1, ..., I_n]$. Can W, the complex Weyl group, act on $\mathcal{M}_\lambda$? This is Coherent Continuation.

Suppose we have a representation π, and we have its character $\theta_\pi = \frac{\sum a(\Lambda)\Lambda(g)}{D(g)}$. How can we act on this character by $w \in W$? We could try

$$\theta_{w\pi}(g) := \frac{\sum a(\Lambda)\Lambda(wg)}{D(wg)}$$

The problem is that W doesn't act on $H(\mathbb{R})$. Another thing to try is

$$\theta_{w\pi}(g) := \frac{\sum a(\Lambda)(w\Lambda)(g)}{D(g)}$$

But W doesn't act on $H(\mathbb{R})$, so how should it act on Λ?

Definition 4.5. Let Λ be a character of $H(\mathbb{R})$, with $d\Lambda = \lambda$ regular and integral. Then $w\lambda - \lambda = \sum n_\alpha \alpha$ is a sum of roots. Define

$$(w \times \Lambda)(h) := \Lambda(h)e^{w\lambda - \lambda}(h)$$

Example: Suppose $H(\mathbb{R}) = \{(e^{i\theta}, x)\} \cong S^1 \times \mathbb{R}^*$. Suppose Λ is the character $\Lambda(e^{i\theta}, x) = e^{2i\theta}|x|$ and $H(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$ with $w(z_1, z_2) = (z_2, z_1)$, for $(z_1, z_2) \in H(\mathbb{C})$. Then $w\lambda - \lambda = (1, 2) - (2, 1) = (-1, 1)$, which corresponds to the character $\frac{z_2}{z_1}$. Then

$$(w \times \Lambda)(e^{i\theta}, x) = (e^{2i\theta}|x|)(\frac{x}{e^{i\theta}}) = e^{i\theta}x|x|$$

Note that we could have done all of the above with the ρ-cover tori, but we chose not to here.

So our new guess for a character formula acted on by W is

$$\theta_{w\pi}(h) = \frac{\sum a(\Lambda)(w \times \Lambda)(h)}{D(h)}$$

Well, this is still a problem for the following reason. Suppose you have a discrete series character formula. Then you have $\theta_\pi(h) = \frac{\sum \epsilon(w)(w\Lambda)(h)}{D(h)}$, but then $\theta_{y\pi}(h) = \frac{\sum \epsilon(w)(yw\Lambda)(h)}{D(h)}$. But this is a problem, because the y is in the wrong place. It should be wy, not yw.

Definition 4.6. Fix an abstract Cartan, W, $w \in W$. Suppose you have a character formula

$$\theta_\pi(h) = \frac{\sum a(\pi, \Delta^+, \Lambda)\Lambda(\tilde{h})}{D(\tilde{h})}$$

Then define

$$\theta_{y\pi}(h) := \frac{\sum a(\pi, Delta^+, \Lambda)(y^{-1}\tilde{\times}\Lambda)(\tilde{h})}{D(\tilde{h})}$$

where $y^{-1}\tilde{\times}\Lambda := w_\Lambda y^{-1} w_\Lambda^{-1} \times \Lambda$, where w_Λ^{-1} is Δ^+-dominant.

Lemma 4.7. *$a(y\pi, \Delta^+, \Lambda) = a(\pi, \Delta^+, wyw^{-1} \times \Lambda)$, where $w^{-1}(d\Lambda)$ is Δ^+-dominant.*

Theorem 4.8. *(Schmid) $y\pi$ is a virtual character.*

Question: Can we do all of this for p-adic groups?

Theorem 4.9. *Suppose we have a representation π, and an H, Λ, Δ^+. Then*

$$a(\pi, \Delta^+, \Lambda) = (-1)^{q_M} M(I(w^{-1} \times \Lambda), w\pi)$$

where $\times$ and $w\pi$ are with respect to Δ^+, and where $w^{-1}(d\Lambda)$ is Δ^+-dominant.

Example: $a(I(H', \Lambda'), \Delta^+, \Lambda) = (-)M(I(w^{-1} \times \Lambda), wI(H', \Lambda'))$. This is computable from the block command in atlas (you don't need KLV polynomials).

Recall the picture from last time:

$\pi(1,+)$ 0 0 1 0 1 0
$\pi(1,-)$ 0 0 0 1 0 1
π_1^+ -1 0 0 0 1 0
π_1^- 0 1 0 0 1 0
where the second through seventh columns give you the coefficients of $e^{i\theta}, e^{-i\theta}, x, |x|, 1/x, 1/|x|$, respectively, in the character formulas of $\pi(1,+), \pi(1,-), \pi_1^+$, and π_1^-. THe first column denoted 4 standard modules I.

We also have a picture

$\mathbb{C}$ 1 -1 1 0 -1 0
$\pi(1,-)$ 0 0 0 1 0 1
π_1^+ -1 0 0 0 1 0
π_1^- 0 1 0 0 1 0
where the first column denotes the irreducible subquotient π of the corresponding standard module I from the previous picture. Note that the representations $\pi(1,-), \pi_1^+, \pi_1^-$ are irreducible already. We can add a 7th column to both pictures, given by $s_\alpha \times I$ in the first picture, and $s_\alpha \times \pi$ in the second picture. In the second picture, this column reads

$s_\alpha\mathbb{C} = \mathbb{C}$

$s_\alpha \pi = \pi$
$s_\alpha \pi_1^+ = \pi_1^+ + \mathbb{C}$
$s_\alpha \pi_1^- = \pi_1^- + \mathbb{C}$

5 Lecture on Character Theory on 07/22/10 at 9AM

Recall:

Have G, K, θ, $G(\mathbb{R})$.

1) Given (H, Δ^+), π, we have a character formula

$$\theta_\pi(h) = \frac{\sum a(\pi, \Delta^+, \Lambda)\Lambda(\tilde{h})}{D(\Delta^+, \tilde{h})}$$

2) We know the characters of $I(H, \Lambda)$.

3) We also know that $a(I(H, \Lambda), \Delta^+, \Lambda') =$ (some sign or zero), on this H, where Λ' is an arbitrary character of the Cartan.

4) We talked about Coherent continuation. Meaning, we have an abstract Weyl group W and we defined what it means for a Weyl group element to act on a distribution character.

Theorem 5.1. *$a(\pi, \Delta^+, \Lambda) = (-1)^{q_M} M(I(w^{-1} \times \Lambda), w\pi)$, where $w^{-1}d\Lambda$ is Δ^+-dominant.*

Atlas copmutes the right hand side, so therefore we can compute the constants $a(\pi, \Delta^+, \Lambda)$. THat is, if $\pi = I$ is a standard module, then just by using the block command, you can compute wI. This is very easy (we will explain this shortly). If π is an irreducible representation, then in order to compute $w\pi$, you need the full computation of KLV polynomials, done by the command wgraph. Or, you can use klbasis to compute π as a sum of standards and then use block to finish off the deal.

So, if we want to compute $M(I, w\pi)$:
Well, if π is a standard module, write $w\pi$ as a sum of standards, and you're done.
If π is irreducible, then write $w\pi$ as a sum of irreducibles, then use klbasis, then we get $M(I, w\pi)$.
Or, if π is irreducible, write it as a sum of standards with the klbasis command, and then you're done.

So, this all answers David Vogan's original question from the first day, the beginning of the notes. In particular, KLV will compute the $a(\pi, \Delta^+, \Lambda)$, which will thus compute the character tables of groups.

Lemma 5.2. *Suppose $I = I(\gamma)$ is a standard module.*

1) If α is a simple root and is not imaginary, then $s_\alpha I(\gamma) = I(s_\alpha \times \gamma)$, where if $\gamma = (H, \Lambda)$, then $s_\alpha \times \gamma := (H, s_\alpha \times \Lambda)$.

2) If α is a simple root and is imaginary, then there are two cases:

i) If α is compact, then $s_\alpha I(\gamma) = -I(\gamma)$

ii) If α is not compact, then $s_\alpha I(\gamma) = -I(s_\alpha \times \gamma) + I(c^\alpha(\gamma))$, where c^α is the Cayley transform (haven't talked about this yet) which is 1 or 2 valued.

Note: The action given by $\times$ everywhere is, in atlas terminology, the cross action.

Lemma 5.3. *Suppose $G(\mathbb{R})$ is connected and π is a discrete series representation with infinitesimal character ρ. Consider the following virtual character*

$$\sum_{w \in W} sgn(w) w\pi$$

where W is the complex Weyl group. Well, you get

$$\sum_{w \in W} sgn(w) w\pi = \pm |W(G(\mathbb{R}), H(\mathbb{R}))| \ \mathbb{C}$$

Example: For a different example, consider $\widetilde{Sp}(2n, \mathbb{R})$, and ω the oscillator representation. Then

$$\theta_\omega(\tilde{g}) \approx \frac{D_{SO}(\tilde{g})}{D_{Sp}(\tilde{g})}$$

where D_{SO} is the Weyl denominator for $SO(2n+1)$ and D_{Sp} is the Weyl denominator for $Sp(2n, \mathbb{R})$. Moreover,

$$|\theta_\omega(g) = \frac{1}{|det(1-g)|^{1/2}}$$

Example: Let F be finite. Jeff thinks he heard that if π is a unipotent representation, then $\theta_\pi(t) = \epsilon_T = \pm 1$ for all regular $t \in T$ on any Cartan T.

Question/Research problem : What can you say about the characters of unipotent representations of real groups? Is there an analogue of what happens in the finite field case? Use atlas to study examples.